Chapter 5

Sentential Logic: Derivations

5.1 – The Derivation System SD

- When evaluating an argument we often try to show that its conclusion can be deduced or derived from its premises.
- Semantics is concerned with the interpretation of a language, i.e., with truth-evaluation.
- Syntax is concerned with the formal properties of a language
- Rules are syntactic when they are applied on the basis of the forms, rather than the truth-values or truth conditions, of sentences.
- Syntactic rules for deriving sentences are called derivation rules.

5.1 – The Derivation System SD

- A derivation rule tells us that, given a group of symbols with a certain structure, we can write down another group of symbols with a certain structure.
- Derivation rules are used to construct derivations that show, in a finite number of steps, how sentences are derived from others.

Defn: a *natural deduction system* (SD) is a system that employs sentential derivation rules.

5.1 – The Derivation System SD

The Rules

1. Reiteration (R)

| ***P***

▸ | ***P***

where ▸ indicates the sentence which is *derived*

1.	$(H \equiv K)\ \&\ S$	A
2.	$\underline{G \vee \sim H}$	A
3.	$(H \equiv K)\ \&\ S$	1 R

where the vertical line is called the *scope line* and the horizontal line marks off the assumptions

5.1 – The Derivation System SD

2. Conjunction Introduction (&I)

$$\begin{array}{l|l} & \boldsymbol{P} \\ & \boldsymbol{Q} \\ \triangleright & \boldsymbol{P} \,\&\, \boldsymbol{Q} \end{array}$$

Note: P and Q do not have to appear one right after the other.

5.1 – The Derivation System SD

3. Conjunction Elimination (&E)

| $\boldsymbol{P} \,\&\, \boldsymbol{Q}$ |
▸ $\boldsymbol{P}$

| $\boldsymbol{P} \,\&\, \boldsymbol{Q}$ |
▸ $\boldsymbol{Q}$

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1)	P & C	A
2)	T & M	A
3)	E & (I & R)	A
4)	P	1 &E
5)	T	2 &E
6)	P & T	4,5 &I
7)	I & R	3 &E
8)	R	7 &E
9)	(P & T) & R	6,8 &I

5.1 – The Derivation System SD

4. Conditional Elimination (⊃E)

$P \supset Q$

P

▸ Q

5.1 – The Derivation System SD

E.g., derive: W & ~F

1)	L & W	A
2)	L ⊃ ~F	A
3)	W	1 &E
4)	L	1 &E
5)	~F	2,4 ⊃E
6)	W & ~F	3,5 &I

5.1 – The Derivation System SD

5. Conditional Introduction (⊃I)

| $\boldsymbol{P}$ (assumption)
| $\boldsymbol{Q}$

▸ $\boldsymbol{P} \supset \boldsymbol{Q}$

5.1 – The Derivation System SD

E.g., derive: $E \supset F$

1)	$E \supset P$	A} primary assumption
2)	$\underline{P \supset F}$	A} primary assumption
3)	$\underline{E}$	A} auxiliary assumption
4)	P	1,3 $\supset$E
5)	F	2,4 $\supset$E
6)	$E \supset F$	3-5 $\supset$I

5.1 – The Derivation System SD

- To derive $\boldsymbol{P} \supset \boldsymbol{Q}$ a *subderivation is* constructed that has $\boldsymbol{P}$ as the assumption and $\boldsymbol{Q}$ as the sentence on the last line.
- When a subderivation is terminated (indicated by ending the scope line), the assumption of the subderivation is said to be discharged, after which none of the lines of the subderivation can be used again.

5.1 – The Derivation System SD

E.g., derive: $G \supset (H \supset K)$

1)	$(G \& H) \supset K$	A
2)	G	A
3)	H	A
4)	$G \& H$	2,3 &I
5)	K	1,4 ⊃E
6)	$H \supset K$	3-5 ⊃I
7)	$G \supset (H \supset K)$	2-6 ⊃I

5.1 – The Derivation System SD

6. Negation Introduction (~I)

$$
\begin{array}{l|l}
 & \underline{\boldsymbol{P}} \\
 & \boldsymbol{Q} \\
 & \sim\boldsymbol{Q} \\
\hline
\triangleright & \sim\boldsymbol{P}
\end{array}
$$

where Q and ~Q can be derived in either order, though both must occur in the subderivation.

5.1 – The Derivation System SD

E.g., derive: ~U

1)	(U & M) ⊃ S	A
2)	M & ~S	A
3)	U	A
4)	M	2 & E
5)	U & M	3,4 &I
6)	S	1,5 ⊃E
7)	~S	2 &E
8)	~U	3-7 ~I

5.1 – The Derivation System SD

7. Negation Elimination (~E)

| | ***~P***
| |---
| | ***Q***
| | ***~Q***

▸ | ***P***

5.1 – The Derivation System SD

E.g., derive: S

1)	$\sim S \supset O$	A
2)	$O \supset S$	A
3)	$\sim S$	A
4)	O	$1,3 \supset E$
5)	S	$2,4 \supset E$
6)	$\sim S$	3 R
7)	S	3-6 $\sim E$

5.1 – The Derivation System SD

8. Disjunction Introduction (∨I)

$$\boldsymbol{P}$$

$$\triangleright\ \boldsymbol{P} \vee \boldsymbol{Q}$$

5.1 – The Derivation System SD

E.g., derive: $M \vee I$

1)	$B \,\&\, E$	A
2)	$[E \vee (F \,\&\, C)] \supset I$	A
3)	E	1 &E
4)	$E \vee (F \,\&\, C)$	3 $\vee$I
5)	I	2,4 $\supset$E
6)	$M \vee I$	5 $\vee$I

5.1 – The Derivation System SD

9. Disjunction Elimination (v E)

$P \vee Q$

P

R

Q

R

▶ R

5.1 – The Derivation System SD

E.g., derive: M & (S ∨ N)

1)	R ∨ V	A
2)	R ⊃ (M & S)	A
3)	V ⊃ (M & N)	A
4)	R	A
5)	M & S	2,4 ⊃E
6)	S	5 &E
7)	S ∨ N	6 ∨I
8)	M	5 &E
9)	M & (S ∨ N)	7,8 &I

5.1 – The Derivation System SD

E.g., derive: M & (S ∨ N)

10)	V	A
11)	M & N	3,10 ⊃E
12)	N	11 &E
13)	S ∨ N	12 ∨I
14)	M	11&E
15)	M & (S ∨ N)	13,14 &I
16)	M & (S ∨ N)	1,4-9,10-15 ∨E

5.1 – The Derivation System SD

10. Biconditional Elimination (≡E)

	$P \equiv Q$				$P \equiv Q$
	P	*or*			Q
▸	Q			▸	P

5.1 – The Derivation System SD

E.g., derive: G

1)	$S \& (S \supset E)$	A
2)	$C \equiv E$	A
3)	$\underline{C \equiv G}$	A
4)	S	1 &E
5)	$S \supset E$	1 &E
6)	E	4,5 $\supset$E
7)	C	2,6 $\equiv$E
8)	G	3,7 $\equiv$E

5.1 – The Derivation System SD

11. Biconditional Introduction (≡I)

$$\begin{array}{l|l} & \begin{array}{|l} \underline{\boldsymbol{P}} \\ \boldsymbol{Q} \end{array} \\ & \begin{array}{|l} \underline{\boldsymbol{Q}} \\ \boldsymbol{P} \end{array} \\ \blacktriangleright & \boldsymbol{P} \equiv \boldsymbol{Q} \end{array}$$

5.1 – The Derivation System SD

E.g., Derive $A \equiv (\sim B \,\&\, \sim N)$

1)	$A \supset (\sim B \,\&\, \sim N)$	A
2)	$\sim B \supset (\sim N \supset A)$	A
3)	A	A
4)	$\sim B \,\&\, \sim N$	1,3 ⊃E
5)	$\sim B \,\&\, \sim N$	A
6)	$\sim B$	5 &E
7)	$\sim N \supset A$	2,6 ⊃E
8)	$\sim N$	5 & E
9)	A	7,8 ⊃E
10)	$A \equiv (\sim B \,\&\, \sim N)$	3-4, 5-9 ≡I

5.2 – Basic Concepts of SD

- A *derivation in SD is a series* of sentences of SL in which each sentence is either taken as an assumption with an indication of its scope or is justified by one of the rules of SD.
- *Scope lines* show immediately which sentences and subderivations are in the scope of which assumption.

5.2 – Basic Concepts of SD

E.g., derive D ∨ B:

1)	~F ∨ D	A
2)	F	A
3)	J	A ← 'J' was not used.
4)	~F	A
5)	~D	A
6)	~F	4 R
7)	F	2 R
8)	D	5-7 ~E *

5.2 – Basic Concepts of SD

E.g., derive $D \vee B$:

9)			<u>D</u>	A
10)			D	9 R
11)		D	1,4-8,9-10 $\vee$E
12)	$D \vee B$	11 $\vee$I

* We could not write $D \vee B$ here, since it would not occur in the scope line of the primary assumption.

5.2 – Basic Concepts of SD

Defn: a sentence ***P*** of SL is *derivable* in SD from a set of sentences Γ of SL iff there is a derivation in SD in which all the primary assumptions are members of Γ and ***P*** occurs in the scope of only those assumptions; we express this by $\Gamma \vdash \boldsymbol{P}$.

Note: not every member of Γ need be taken as a primary assumption.

5.2 – Basic Concepts of SD

Defn: an argument of SL is *valid in SD* iff the conclusion of the argument is derivable in SD from the set consisting of the premises. An argument of SL is invalid in SD iff it is not valid in SD.

E.g.,

$(\sim L \vee K) \supset A$

$\underline{A \supset \sim A}$

$L \,\&\, \sim K$

5.2 – Basic Concepts of SD

1)	$(\sim L \vee K) \supset A$	A
2)	$A \supset \sim A$	A
3)	$\sim L$	A
4)	$\sim L \vee K$	3 $\vee$I
5)	A	1,4 $\supset$E
6)	$\sim A$	2,5 $\supset$E
7)	L	3-6 $\sim$E
8)	K	A
9)	$\sim L \vee K$	8 $\vee$I
10)	A	1,9 $\supset$I
11)	$\sim A$	2,10 $\supset$E
12)	$\sim K$	8-11 $\sim$I
13)	$L \,\&\, \sim K$	7,12 & I

5.2 – Basic Concepts of SD

Defn: a sentence ***P*** of SL is a *theorem in SD* iff P is derivable in SD from the empty set, i.e., iff ***P*** requires no primary assumptions.

E.g., derive: $B \supset [C \supset (B \& C)]$

1)	B	A
2)	C	A
3)	$B \& C$	1,2 &I
4)	$C \supset (B \& C)$	2-3 $\supset$I
5)	$B \supset [C \supset (B \& C)]$	1-4 $\supset$I

5.2 – Basic Concepts of SD

The claim that a sentence P of SL is a theorem in SD is expressed as $\varnothing \vdash \boldsymbol{P}$ or just $\vdash \boldsymbol{P}$.

Defn: sentences ***P*** and ***Q*** of SL are *equivalent in SD* iff ***Q*** is derivable in SD from {***P***} and ***P*** is derivable in SD from {***Q***}.

E.g., $\boldsymbol{P}$ = (G & S) ∨ N and $\boldsymbol{Q}$ = (G ∨ N) & (S ∨ N)

5.2 – Basic Concepts of SD

i) derive (G ∨ N) & (S ∨ N):

1)	(G & S) ∨ N	A
2)	G & S	A
3)	G	2 &E
4)	G ∨ N	3 ∨I
5)	S	2 &E
6)	S ∨ N	5 ∨I
7)	(G ∨ N) & (S ∨ N)	4,6 &I
8)	N	A
9)	G ∨ N	8 ∨I
10)	S ∨ N	8 ∨I
11)	(G ∨ N) & (S ∨ N)	9,10 &I
12)	(G ∨ N) & (S ∨ N)	1,2-7,8-11 ∨E

5.2 – Basic Concepts of SD

ii) derive (G & S) ∨ N:

1)	(G ∨ N) & (S ∨ N)	A
2)	G ∨ N	1 &E
3)	S ∨ N	1 &E
4)	N	A
5)	(G & S) ∨ N	4 ∨I
6)	G	A
7)	S	A
8)	G & S	6,7 &I
9)	(G & S) ∨ N	8 ∨I
10)	(G & S) ∨ N	3,7-9, 4-5 ∨E
11)	(G & S) ∨ N	2,6-10,4-5 ∨E

5.2 – Basic Concepts of SD

Defn: a set of sentences of SL is *inconsistent in SD* iff both a sentence ***P*** of SL and ~***P*** are derivable in SD from Γ.

Defn: A set of sentences of SL is *consistent* in SD iff it is not inconsistent in SD.

E.g., show {(M ∨ B) ⊃ B, A ⊃ M, A & ~B} is inconsistent in SD.

5.2 – Basic Concepts of SD

1)	$(M \vee B) \supset B$	A
2)	$A \supset M$	A
3)	$\underline{A \& \sim B}$	A
4)	$\sim B$	3 &E
5)	A	3 &E
6)	M	2,5 $\supset$E
7)	$M \vee B$	6 $\vee$I
8)	B	1,7 $\supset$E

5.2 – Basic Concepts of SD

*Notice that anything follows from an inconsistent set:

1)	P		
2)	~P		
3)		~Q	A
4)		P	1R
5)		~P	2 R
6)	Q		3-4 ~E

5.2 – Basic Concepts of SD

The connections between semantic and syntactic concepts:

1) A sentence ***P*** is derivable in SD from a set Γ of sentences of SL iff ***P*** is truth-functionally entailed by Γ.
2) An argument of SL is valid in SD iff the argument is truth-functionally valid.
3) A sentence ***P*** of SL is a theorem in SD iff it is truth-functionally true.
4) Sentences ***P*** and ***Q*** of SL are equivalent in SD iff ***P*** and ***Q*** are truth-functionally equivalent.
5) A set Γ of sentences of SL is inconsistent in SD iff Γ is truth-functionally inconsistent.

5.3 – Strategies for Constructing Derivations in SD

All rules apply to (a) entire sentences and (b) to entire subderivations on earlier lines.

a) e.g., derive: K ∨ G

1.	(~J & W) & Y	A
2.	<u>(N ∨ ~B) & (~J ⊃ (K ∨ G))</u>	A
3.	~J	1 &E←

Error! &E must be applied to (~J & W)

5.3 – Strategies for Constructing Derivations in SD

derive: K ∨ G

1.	(~J & W) & Y	A
2.	(N ∨ ~B) & [~J ⊃ (K ∨ G)]	A
3.	~J & W	1 &E
4.	~J	3 &E
5.	K ∨ G	2,4⊃E←

Error! ⊃E can only be applied where the main connective is ⊃.

5.3 – Strategies for Constructing Derivations in SD

b) e.g., derive: ~N

1.	H ⊃ ~N	A
2.	(H ∨ G) & ~M	A
3.	~N ≡ (G ∨ B)	A
4.	H ∨ G	2 &E
5.	H	A
6.	~N	1,5 ⊃E
7.	~N ∨ H	6 ∨I ← Line 7 not needed.
8.	G	A
9.	G ∨ B	8 ∨I
10.	~N	3,9 ≡E
11.	~N	4,5-6,8-10∨E ← Error!

5.3 – Strategies for Constructing Derivations in SD

c) once an assumption has been discharged, none of the lines or subderivations within that subderivation can be used to justify sentences on other lines.

E.g.,

1.	$\sim U \supset \sim W$	A
2.	$\sim W \supset \sim S$	A
3.	$\quad \sim U$	A
4.	$\quad \sim W$	1,3 $\supset$E
5.	$\quad \sim S$	2,4 $\supset$E
6.	$\sim U \supset \sim S$	3-5 $\supset$I
7.	$\sim S$	2,4 $\supset$E ←

Error! Line 4 can no longer be appealed to.

5.3 – Strategies for Constructing Derivations in SD

- Write down the goal sentence well below any primary assumption.
- Use goal analysis: a) work backwards from goal sentence, b) consider sentences available on earlier lines, and c) create subgoals.

i) What is its main connective? What are its component sentences?

ii) What is its main connective? Are any components similar to those of the (sub)goal sentence? Can the goal be directly derived? What are the possible subgoals?

5.3 – Strategies for Constructing Derivations in SD

iii) Identify a derivation (subgoal) from which you can see how to reach the goal sentence. Select an introduction or elimination subgoal. Regard the subgoal as the 'new' goal. Repeat the process.

- Subgoals for introduction rules and elimination rules are listed in the text.

5.4 – The Derivation System SD+

- SD+ contains all the derivation rules of SD plus some more. SD+ is not a stronger system than SD.
- A sentence is derivable in SD+ from a set of sentences of SL *iff* it is derivable in SD from a set of sentences of SL.
- SD+ is a *conservative extension* of SD.

5.4 – The Derivation System SD+

Rules of Inference

Modus Tollens (MT)

$P \supset Q$
$\sim Q$
$\blacktriangleright \sim P$

Hypothetical Syllogism (HS)

$P \supset Q$
$Q \supset R$
$\blacktriangleright P \supset R$

Disjunctive Syllogism (DS)

$P \vee Q$
$\sim P$
$\blacktriangleright Q$

or

$P \vee Q$
$\sim Q$
$\blacktriangleright P$

5.4 – The Derivation System SD+

1)	**P ⊃ Q**	A
2)	**Q ⊃ R**	A
3)	**P**	A
4)	**Q**	1,3 ⊃E
5)	**R**	2,4 ⊃E
6)	**P ⊃ R**	3-5 ⊃I

Hypothetical Syllogism allows skipping lines (3) through (5) in a derivation

5.4 – The Derivation System SD+

Rules of Replacement

Double Negation (DN)

$P \leftrightarrow \sim\sim P$

Transposition (Trans)

$P \supset Q \leftrightarrow \sim Q \supset \sim P$

Implication (Impl)

$P \supset Q \leftrightarrow \sim P \vee Q$

De Morgan (DeM)

$\sim(P \,\&\, Q) \leftrightarrow \sim P \vee \sim Q$

$\sim(P \vee Q) \leftrightarrow \sim P \,\&\, \sim Q$

5.4 – The Derivation System SD+

Exportation (Exp)

$\boldsymbol{P} \supset (\boldsymbol{Q} \supset \boldsymbol{R}) \triangleleft\triangleright (\boldsymbol{P} \;\&\; \boldsymbol{Q}) \supset \boldsymbol{R}$

Commutation (Com)

$\boldsymbol{P} \;\&\; \boldsymbol{Q} \triangleleft\triangleright \boldsymbol{Q} \;\&\; \boldsymbol{P}$

$\boldsymbol{P} \vee \boldsymbol{Q} \triangleleft\triangleright \boldsymbol{Q} \vee \boldsymbol{P}$

Association (Assoc)

$\boldsymbol{P} \;\&\; (\boldsymbol{Q} \;\&\; \boldsymbol{R}) \triangleleft\triangleright (\boldsymbol{P} \;\&\; \boldsymbol{Q}) \;\&\; \boldsymbol{R}$

$\boldsymbol{P} \vee (\boldsymbol{Q} \vee \boldsymbol{R}) \triangleleft\triangleright (\boldsymbol{P} \vee \boldsymbol{Q}) \vee \boldsymbol{R}$

Idempotence (Idem)

$\boldsymbol{P} \triangleleft\triangleright \boldsymbol{P} \;\&\; \boldsymbol{P}$ $\qquad$ $\boldsymbol{P} \triangleleft\triangleright \boldsymbol{P} \vee \boldsymbol{P}$

5.4 – The Derivation System SD+

Distribution (Dist)

$$P \,\&\, (Q \vee R) \blacktriangleleft\!\blacktriangleright (P \,\&\, Q) \vee (P \,\&\, R)$$

$$P \vee (Q \,\&\, R) \blacktriangleleft\!\blacktriangleright (P \vee Q) \,\&\, (P \vee R)$$

Equivalence (Equiv)

$$P \equiv Q \blacktriangleleft\!\blacktriangleright (P \supset Q) \,\&\, (Q \supset P)$$

$$P \equiv Q \blacktriangleleft\!\blacktriangleright (P \,\&\, Q) \vee ({\sim}P \,\&\, {\sim}Q)$$

5.4 – The Derivation System SD+

- Rules of Inference (*MT*, *HS*, *DS*) apply to entire sentences on a line of a derivation.
- Rules of Replacement can apply to entire sentences or to sentences that are parts of other sentences.
- E.g. P & Q ▸ ~~(P & Q)

  P & Q ▸ P & ~~Q

  are both permissible uses of Double Negation

5.4 – The Derivation System SD+

1)	(A v ~B) v ~C	A
2)	(D v G) v C	A
3)	(~~A v ~B) v ~C	1DN
4)	~(~A & B) v ~C	3 DeM
5)	(~A & B) ⊃ ~C	4 Impl
6)	C v (D v G)	2 Com
7)	C v (G v D)	6 Com
8)	~~C v (G v D)	7 DN
9)	~C ⊃ (G v D)	8 Impl
10)	(~A & B) ⊃ (G v D)	5,9 HS
11)	~A ⊃ [B ⊃ (G v D)]	10 Exp