# CHAPTER 8 – PREDICATE LOGIC: SEMANTICS

# SECTION 8.1 INTERPRETATIONS

- In SL, the basic semantic concept was truth-value assignments.
- But PL is more complex, because its atomic sentences and quantified sentences can themselves be complex.

- E.g. 'Fa' is composed of a predicate and an individual constant.

- Basic *semantic* concept of PL: the *interpretation*. Every individual constant, predicate, and sentence letter of the language is interpreted, though in practice we shall confine our attention to those which are relevant to the problem at hand.
- [Strictly speaking, a language is interpreted, but we are interested only in the interpretation of sentences.]

# SECTION 8.1 INTERPRETATIONS

- Symbolization keys for sentences embody interpretations for them. The elements are the UD and the assignments made to the predicates and the constants.
- The truth conditions of sentences of PL depend on our selected universe of discourse, UD, which can be any nonempty set (e.g. natural number, people, etc.).

- E.g. Whether `Fa' is true depends on how we interpret `F' and 'a' relative to a UD.
- So, if `Fx' is interpreted as `x is human' and `a' is interpreted as `Socrates', then `Fa' is true (assuming our UD is all living things).
- But `Fa' is false if we instead interpret `Fx' as `x is a potato'. When the predicate `F' fails to pick out something in our UD, e.g. `x is a rock', then its extension is the empty set.

# SECTION 8.1 INTERPRETATIONS

- *Definition:* An *interpretation* of PL specifies an nonempty set as a UD and assigns a truth-value to each sentence letter of PL, a member of the UD to each individual constant of PL, and a set of **n**-tuples of members of the UD to each n-place predicate of PL.

- Consider two-place predicates. Suppose UD: Set of positive integers; Gxy: x is greater than y.

- Here, `G' is a *two place* predicate, so its *extension* is a set of *ordered pairs* of objects.  E.g. the pair 5 and 2 (in that order).

- Three-place predicates, four-place predicates, and so on, are all interpreted similarly.  E.g. a three-place predicate is a set of ordered triples.

- For a one-place predicate, the extension is a subset of the UD. If no member of the UD satisfies the predicate, then it has the empty set as its extension given the interpretation.

- Predicates of more than one place are relational and have as their extensions sets of *n*-tuples drawn from the UD. In the example, the predicate 'G' is two-place, and so its extension is a set of ordered pairs.

- The reason for the ordering requirement is given through a case where a predicate is satisfied by one ordering but not by the reverse ordering. In some cases, the relation is reflexive or symmetrical, in which case the ordering makes no difference.

- Once we have an interpretation, we can then determine the truth-value of a sentence on that interpretation.

- A sentence is *true* on an interpretation just in case the object picked out by a constant is a member of the set that is the extension of the relevant predicate.

- E.g. UD: The set of things in this classroom; Fx: is a piece of furniture; a: the chair with the most ink marks on it.

- `Fa' is true just in case `a' designates a member of the extension of `F', which it does in this case.

- `Fa' is false just in case `a' fails to designate a member of the extension of `F' , which it would, if `Fx' was interpreted as `x is a table', or, if `a' was interpreted as `the window closest to the front'.

- The interpretation of constants is by reference to (designation of) a single member of the selected UD. This can change from interpretation to interpretation, and the resulting truth-values may change as well.

- A single constant may have only one referent, but many constants can share the same referent.

- E.g. We allow the following.  UD: planets; a: Uranus; b: Saturn.

- E.g. We DO NOT allow the following.  UD: planets; c: Saturn; c; Jupiter.

- For compound sentences of PL that do not contain quantifiers,  the truth-conditions are determined in accordance with the truth-functional connectives in the usual way.

- Consider the sentence `(Bs v ~ Fh) & Gsh'.  Depending on our interpretation, this sentence may be either true or false.

- It is true on the following interpretation.  UD: Set of people; Bx: x is a male; Fx: x is a negative integer; Gxy: x is the mother of y; h: Jay doe; s: Jane Doe (Jay Doe's mother).

- It is false on the following interpretation.  UD: Set of all things; Bx: x is an author; Fx: x is an animal; Gxy: x owns y; h: The Liberty Bell; s: Stephen King.

- Things are different when we have quantifiers.  Quantified sentences are not atomic nor truth-functions of smaller sentences. Variables do not function like names, so an interpretation does not assign them members of the UD. That is, variables are not interpreted. Rather, variables are more like pronouns; each is treated as if it stands for any member of the UD.

- E.g. `$(\forall x)(Fx \supset Gx)$' may be read as `Each x is such that if x is F then x is G'or ' All F are G'.

- Once we specify a UD, then we know what 'each x' is. E.g. if UD: Set of people, then 'each x' is just every person.

- So, a formula like 'Fx $\supset$ Gx' specifies a condition, and then a universal quantifier says that this condition holds for each member of the UD.

- E.g. UD: Set of people; Fx: x is a politician; Gx: x is honest. Here, `($\forall$x)(Fx $\supset$ Gx)' means `Every politician is honest', which is false on this interpretation, since at least one politician is not honest.

- Universally quantified sentences are false if there is at least one member of the UD that does not hold of the specified condition that follows the quantifier.

- Recap: variables are not interpreted, they range over the members of the UD.  Consequently, the truth-conditions for a quantified sentence of PL depends on two things: the selected UD for an interpretation, and the individual variables and quantifies.

- Universal quantifiers say that every member of the UD meets the specified condition that lies in its scope.

- Existential quantifiers say that at least one member of the UD meets the specified condition that lies in its scope.

- A universally quantified sentence is true iff all members of the UD satisfy the condition specified by the relevant open sentence.

- An existentially quantified sentence is true iff at least one member of the UD satisfies the condition specified by the relevant open sentence.

- Consider `(∃x)(Cx & Bx)'.  This sentence is false on the following interpretation. UD: Set of all things; Cx: x is a car; Bx: x has a brain.

- But the sentence `(∃x)(Cx) & (∃x)(Bx)' is true on the same interpretation.  This is because the scope of the quantifiers are different.

- Let UD: Set of people; Fxy: x is acquainted with y.  Consider `($\exists$x)($\forall$y) Fxy' which says `there is someone that is acquainted with everyone'. This sentence is false.

- Consider the sentence where the quantifiers are reversed, i.e., `($\forall$y)($\exists$x) Fxy'.  This sentence says `everyone is acquainted with someone'. This sentence is likely true.

- Let UD: Set of houses in the world; Fx: x is made of brick; Gxy: x is larger than y.  Then `$((\forall x)Fx \supset (\exists y) \sim Gyy)$' says ` If every house is made of brick, then there is some house which is not larger then itself'.

- Remember, if some object does not satisfy the antecedent, it nevertheless *trivially* satisfies the entire condition. So for any house that is not made of brick, the above sentence is still true of it.

# SECTION 8.1 INTERPRETATIONS

- Recap: truth-conditions for sentences in PL are determined by interpretations.

- An interpretation is a UD and the interpretation of all sentence letters, predicates, and individual constants of PL. However, as we have done so far, we've only interpreted the relevant parts of the language, i.e., the parts that appear in the sentences were interested in.

- *Definition:* A sentence **P** of PL is *true on an interpretation* **I** if and only if every variable assignment satisfies **P** on **I**. A sentence **P** of PL is false on an interpretation **I** if and only if no variable assignment satisfies **P** on I.

- *Definition:* A sentence **P** of *PL* is *quantificationally true* if and only if **P** is true on every interpretation.

- *Definition:* A sentence **P** of *PL* is *quantificationally false* if and only if **P** is false on every interpretation.

- *Definition:* A sentence **P** of *PL* is *quantificationally indeterminate* if and only if **P** is neither quantificationally true nor quantificationally false.

- `$(\exists x)$ (Gx $\lor$ ~ Gx)' is quantificationally true, i.e. it is true on every interpretation.

- Obviously we can't check this on every interpretation, but we needn't do so anyway.

- Without even knowing the interpretation of `G' or what our UD is, we know that every member of any UD satisfies the condition.

- `($\forall$x) Bx & ($\exists$x) ~ Bx' is quantificationally false, i.e., it is false on every interpretation.

- Again, we don't need to think of any particular interpretation. Any interpretation which makes the first conjunct true means that every member of our UD is in the extension of `B', but then no member can satisfy the second condition `~ Bx'.  So one of the conjuncts must always be false, and hence the conjunction is always false.

- To show that a sentence is NOT quantificationally true, we need only find at least one interpretation on which it is false.

- E.g., `($\forall$x) (Fx $\vee$ Gx)' is false under the following interpretation. UD: Set of positive integers; Fx: x is odd; Gx: x is greater than 4. It is false because 2 does not satisfy either condition, and the sentence say that every member of the domain satisfies either one or the other condition.

- To show that a sentence is NOT quantificationally false, we just need to find an interpretation on which it is true.

- A quantificationally indeterminate sentence is one that is neither quantificationally true, nor quantificationally false.
- To show that a sentence is quantificationally indeterminate, we construct one interpretation on which the sentence is false, and one on which the sentence is true.
- E.g. `~ ( ~ Ga & (∃y) Gy )' is true on the following interpretation. UD: Set of positive integers; Gx: x is even; a: 2.
- But it is false on the following interpretation. UD: Set of positive integers; Gx: x is odd; a: 2.

- Why do we keep using the set of positive integers as our UD?

- The Lowenheim theorem says that if a sentence of predicate logic without identity is true on at least one interpretation, then it is true on some interpretation where the UD is the set of positive integers.

- So, even though we could use just about any other UD and predicates, the positive integers serve us well for any sentence we're interested in.

- *Definition:* Sentences **P** and **Q** of *PL* are *quantificationally equivalent* if and only if there is no interpretation on which P and Q have different truth-values.

- E.g. `$(\exists x)\, Fx \supset Ga$' and `$(\forall x)\, (Fx \supset Ga)$'

- Remember, we do not reason about when they are true (or false) on a particular interpretation, rather, we reason about them on some interpretation and see what follows when we assume they are true (or false).

# SECTION 8.3 QUANTIFICATIONAL EQUIVALENCE AND CONSISTENCY

- To establish that two sentences are NOT quantificationally equivalent, we construct an interpretation such that one of our sentences is true on it, and the other is false.
- E.g. A: `($\forall$x) (Fx $\supset$ Ga)' and B: `($\forall$x) Fx $\supset$ Ga' are not quantificationally equivalent on the following interpretation. (Note that A and B are not the same, their quantifiers have different scope.) UD: Set of positive integers;
- Fx: x is prime; Gx: x is even; a: 1.
- The trick for coming up with our interpretation was finding a way to make `Ga' false so that A would come out false, but still have B come out true by ensuring that not everything in our UD was in the extension of `F'.

# SECTION 8.3 QUANTIFICATIONAL EQUIVALENCE AND CONSISTENCY

- It is important to remember that we construct a particular interpretation in order to show that two sentences are NOT quantificationally equivalent, but we CANNOT use the same method when it comes to showing that two sentences ARE quantificationally equivalent.

- To show that two sentences are quantificationally equivalent, we must reason more abstractly and consider what follows when we assume that our sentences are true (or false) on some (but not particular) interpretation.

- *Definition:* A set of sentences $\Gamma$ of PL is *quantificationally consistent* if and only if there is at least one interpretation on which all the members of the set are true.

- A set of sentences of PL is *quantificationally inconsistent* if and only if the set is NOT quantificationally consistent.

- E.g. The set {$(\forall$x) Gax,  ~ Gba $\vee$ ($\exists$x) ~ Gax} is quantificationally consistent because both sentences of the set are true on the following interpretation.

    UD: Set of positive integers

    Gxy: x is less than or equal to y

    a: 1

    b: 2

# SECTION 8.3 QUANTIFICATIONAL EQUIVALENCE AND CONSISTENCY

- A somewhat surprising example of a quantificationally consisten set is the following:

- $\{(\forall w) (Fw \supset Qw), (\forall w) (Fw \supset \sim Gw)\}$.

- The sentences are consistent on an interpretation where no object is in the extension of `F'.

- E.g. UD: Set of positive integers; Fx: x is negative; Gx: x is even.

- So we can show that a set of sentences is quantificationally consistent by constructing a single interpretation, but we CANNOT use the same strategy to show that a set of sentences is quantificationally inconsistent.

- To show that a set of sentences is quantificationally inconsistent, we must reason about them and show that there is no interpretation that could make all the sentences true.

- E.g. { ($\exists$y) (Fy & ~ Ny), ($\forall$y) (Fy $\supset$ Ny) } We know this set is inconsistent because whenever one of these sentences is true on some interpretation, the other sentence must be false on that same interpretation. Hence, there is not interpretation on which both can be true.

# SECTION 8.3 QUANTIFICATIONAL EQUIVALENCE AND CONSISTENCY

- To show that a set of sentences is quantificationally inconsistent, we must reason about them and show that there is no interpretation that could make all the sentences true.

- E.g. { (∃y) (Fy & ~ Ny), (∀y) (Fy ⊃ Ny) } We know this set is inconsistent because whenever one of these sentences is true on some interpretation, the other sentence must be false on that same interpretation.  Hence, there is not interpretation on which both can be true.

- *Definition:* A set $\Gamma$ of sentences of *PL quantificationally entails* a sentence P of *PL* if and only if there is no interpretation on which every member of $\Gamma$ is true and P is false.

- For example, the set $\{(\forall x)(Bx \supset Ga), (\exists x) Bx\}$ quantificationally entails (or q-entails for short) the sentence `Ga'.

- And the set $\{(\forall x)(\sim Jx \lor (\exists z)Kz), (\exists x)(Jy)\}$ q-entails $(\exists z)Kz$.

- In both examples, any interpretation that makes the set of sentences true, also makes `Ga' and `$(\exists z)Kz$' true respectively.

- *Definition:* An argument of *PL* is *quantificationally valid* if and only if there is no interpretation on which every premise is true and the conclusion is false.  An argument and of *PL* is *quantificationally invalid* if and only if the argument is not quantificationally valid.

- E.g. The following argument is valid.

$(\exists x)(Fx \lor Gx)$
$\underline{(\forall x) \sim Fx}$
$(\exists x) Gx$

- On any interpretation where both premises are true, the conclusion must be true as well.

- To show that a set of sentences does NOT q-entail a sentence, we construct an interpretation where the members of the set are true, but the sentence in question is false.

- E.g., $\{\sim(\forall x)(Gx \equiv Fx), \sim Fb\}$ does not q-entail $(\forall x) \sim Gx$ on the following interpretation. UD: Set of positive integers; Fx: x is greater than 5; Gx: x is prime; b:3.

- To show that an argument is quantificationally invalid, we construct an interpretation on which the premises are true and the conclusion is false.
- E.g., the following argument,

(∃x) ( (∃y)Fy ⊃ Fx)
<u>(∃y) ~ Fy</u>
~ (∃x) Fx

is quantificationally invalid because we have the following interpretation in which the premises are true but the conclusion false. UD: Set of positive integers; Fx: x is prime

- Again, remember that we CANNOT prove that a q-entailment holds or that an argument IS quantificationally valid by the construction of a particular interpretation.  Instead, we show q-entailment or quantificiationally validity by reasoning about the truth-values of sentences on every interpretation.

# SECTION 8.5 TRUTH-FUNCTIONAL EXPANSIONS

- Many UD are very large or infinite. The set of positive integers, for example, is infinite.

- The next notion we are introducing, *truth-functional expansions*, allows us to reason about the truth-values of sentences with small UDs.

- Consider UD: The set {1,2}; Wx: x is even; Cxy: x is greater than y.

- The sentence `$(\forall x)(Wx \supset (\exists y)(Cxy)$' is true on this interpretation. This sentence can be replaced with a sentence that has no quantifiers, but still says the same thing about our UD. We do so by designating each member of the UD with a constant, e.g. a:1 and b:2. Then the sentence `$(Wa \supset (Caa \lor Cab))$ & $(Wb \supset (Cba \lor Cbb))$' says the same thing about our UD as the one above. This new sentence is called the *truth-functional expansion* of our original sentence.

# SECTION 8.5 TRUTH-FUNCTIONAL EXPANSIONS

- Generally, whenever we have a sentence with a universal quantifier and a finite UD, such that every member of the UD has a least one constant designating it, then we can re-express the universally quantified sentence as a *conjunction* of its substitution instances formed from the constants. Since every member of the UD has at least one constant designating it, then the conjunction says that every member meets some condition, which is just what the universal quantifier says.

- Almost the exact same holds for existentially quantified sentences, except that it is re-expressed as a *disjunction* of its substitution instances formed from the constants. In this case the sentence would say that at least one member of the UD meets some condition.

# SECTION 8.5 TRUTH-FUNCTIONAL EXPANSIONS

- More precisely, we construct a truth-functional expansion of a universally quantified sentence in the following way. We take the sentence $(\forall x)P$, remove the initial quantifier (to get an open sentence), and then replace the sentence with $(...(P(a_1/x)$ & $P(a_2/x))$ & ... & $P(a_n/x))$, where $a_1, ..., a_n$ are the constants and $P(a_i/x)$ is the substitution instance of $(\forall x)P$.

- E.g. `$(\forall x)$ Nx', with the set of constants {`a', `b'} expands to `Na & Nb', and `$(\forall y)(My \supset Jyy)$' expands to `$(Ma \supset Jaa)$ & $(Mb \supset Jbb)$'. Both the unexpanded and that expanded sentences are true for a UD that has two members (assuming that the two members are named by our constants).

- If we increase our set of constants to {`a', `b', `c'} then `$(\forall x)$ Nx' expands to `Na & Nb & Nc'  and `$(\forall y)(My \supset Jyy)$' expands to  `$(Ma \supset Jaa)$ & $(Mb \supset Jbb)$ & $(Mc \supset Jcc)$'.

- In the case that our sentence has more than one universal quantifier, we proceed from left to right.

- E.g. Expanding `($\forall$y)(Ly & ($\forall$z)Bzy)' for the set of constants {`a', `b'} we first have `(La & ($\forall$z)Bza) & (Lb & ($\forall$z)Bzb)'. Then we eliminate the second quantifier (which now has two occurrences), so that we get `(La & (Baa & Bba)) & (Lb & ($\forall$z)Bzb)' and then `(La & Baa & Bba) & (Lb & (Bab & Bbb))'

- For existentially quantified sentences, we proceed in exactly the same fashion as we did with universally quantified sentences, except that we use iterated *disjunction* instead of iterated conjunction.

- When expanding a sentence, we may choose a set that has only one constant.  In this case, we remove the quantifier and then replace the free variable from the resulting open sentence with the constant in the set.

- E.g., `∀xFx' is expanded as `Fa' for the set of constants {`a'} and `∃xFx' as `Fa'. Similarily, we first expand `(∀x)(Fx ∨ (∃y)Gyy)' as `(Fa ∨ (∃y)Gyy)' and then as `(Fa ∨ Gaa)'.

- Since every expanded sentence of PL eliminates every quantifier, the result is either an atomic sentence or a truth-functional compound of atomic sentences. That means we can construct truth tables, which in turn tell us about the truth-conditions of the sentences.

- Why is this interesting? Because we can use the truth tables to construct interpretations that we know will make the unexpanded sentence true (or false).

- E.g., an expansion of `($\exists$x)($\forall$y)Nyx' for {`a',`b'} is `(Naa & Nba) $\vee$ (Nab & Nbb)'. Here is a shortened truth-table on which the expansion is true (for a UD with two members).

| Naa | Nab | Nba | Nbb | l | (Naa | & | Nba) | $\vee$ | (Nab | & | Nbb) |
|-----|-----|-----|-----|---|------|---|------|--------|------|---|------|
| T | T | F | T | l | T | F | F | T | T | T | T |

- In fact, we need not construct an interpretation.  The truth table guarantees that there is such an interpretation; namely one in which the member designated by `b' fails to stand in the relation N to the member designated by `a', but all other members stand in the relation N to others (including themselves).

- The same goes for guaranteeing that there is an interpretation on which an unexpanded sentence is false.

| Naa | Nab | Nba | Nbb | I | | (Naa | & | Nba) | $\lor$ | (Nab | & | Nbb) |
|-----|-----|-----|-----|---|---|------|---|------|--------|------|---|------|
| F | F | F | F | I | | F | F | F | F | F | F | F |

- E.g., an expansion of `($\exists$x)($\forall$y)Nyx' for {`a',`b'} is `(Naa & Nba) $\lor$ (Nab & Nbb)'. Here is a shortened truth-table on which the expansion is false (for a UD with two members).

- The truth-table ensures us that there is an interpretation on which our unexpanded sentence is false -- one on which no members of UD stand in the relation N to one another (including themselves).

- Since truth-functional expansions provide us with truth-tables with which we can show when a sentence is true or false (in a finite UD), we can use them to show that certain sentences have, or fail to have, certain semantic properties.

- E.g. to show that a sentence is not quantificationally true, we need to find an interpretation on which the sentence is false.  Or, to show that a sentence is not quantificationally false, we need to find an interpretation on which the sentence is true.  Likewise with quantificational equivalence.

- Note: we cannot show that a sentence IS quantificationally true, even if we find that the full truth-table of the expanded sentence is true on ever row of the truth-table.  (The sentence is true on every interpretation that has the same size UD or smaller, but this is not guaranteed for a UD that is larger.)

- E.g., we show that `($\forall$x)(Fx $\supset$ Ga)' and `($\forall$x) Fx $\supset$ Ga' are not quantificationally equivalent by expanding both sentences for a set of constants (which must include `a'), {`a', `b'}: `(Fa $\supset$ Ga) & (Fb $\supset$ Ga)' and `(Fa & Fb) $\supset$ Ga'.  Then we show that they have different truth-values on a least one row in the truth-table (we just give the shortened one here).

| Fa | Fb | Ga | I | (Fa | $\supset$ | Ga) | & | (Fb | $\supset$ | Ga) | | (Fa | & | Fb) | $\supset$ | Ga |
|----|----|----|---|-----|-----------|-----|---|-----|-----------|-----|-|-----|---|-----|-----------|-----|
| T  | F  | F  | I | T   | F         | F   | F | F   | T         | F   | | T   | F | F   | T         | F   |

- The shortened truth-table shows that there is an interpretation on which `($\forall$x)(Fx $\supset$ Ga)'  is false and `($\forall$x) Fx $\supset$ Ga' is true, hence they are not quantificationally equivalent.

- Similarily, we can use truth-functional expansions to show that some sentences of PL are quantificationally consistent, or that some arguments are not quantificationally valid.

- The identity predicate '=' is not explicitly given an interpretation. Its extension is the set of ordered pairs of the members of UD (whatever UD is) where the first member is identical to the second. So *PLE* interpretations are the same as for *PL.*

- E.g. If the UD is the positive integers, then the extension of the identity predicate is determined as all and only the pairs: (1,1), (2,2), (3,3), …

- **a=a** (where **a** is arbitrary) is true on every interpretation. However, the truth value of **a=b (**where **a** and **b** are different constants) depends on their interpretations.

- E.g. `g=k' is true when UD: {PI}; g: 1, and k:1, but `g=k' is false when UD: {PI}, g: 1, and k: 2.

- Sets of sentences containing the identity predicate have (or fail to have) certain semantic properties.

- E.g. $(\forall x)(\forall y)(\sim x=y \lor (Fx \supset Fy))$ is quantificationally true -- either x and y are not identical, or whatever predicate x is in the extension of, so is y. One of these disjuncts is guaranteed to be true, no matter what the UD is.

- E.g. $(\forall x)(\forall y)(x=y \lor (Fx \supset Fy))$ is not quantificationally true, since we have the following interpretation that makes it false. UD: {1,2}; Fx: x is odd.

$(\forall x)(Fx \equiv Gx)$
$(Ax)(\forall y)\ x=y$
Ga _____ .
$(\forall x)Fx$

- The above argument is quantificationally valid because any interpretation that makes the premises true is guaranteed to make the conclusion true. However, the following argument is not quantificationally valid,

$(\forall x)(\exists y)\ x=y$
a=b

- because the conclusion is false on the following interpretation (even though the premise is true). UD: {PI}; a:6; b:7.

- When sentences with the identity predicate are truth-funcationally expanded, one must be slightly more careful with the relevant truth-tables.  Some rows of the truth-table will not correspond to any interpretations, hence cannot be used to establish semantic properties.  The following two conditions must be met for constructing a truth-table for an expansion that contains the identity predicate (which rule out rows that do not have corresponding interpretations):

1. Every sentence of the form **a=a** has the truth-value **T**

2. If a sentence of the form **a=b** has the truth-value **T**, then for each atomic sentence **P** that contains **a**, every atomic sentence **P(b//a)** that results from replacing one or more occurrences of **a** in **P** with **b** must have the truth-value as **P**.

E.g. a sentence with **a=b** that has truth-value **T** on some row will also have **T** for **b=a.**

- Consider the sentence '$(\forall x)(Px \supset Hf(x))$'. '*f*' is a one-place functor (two-place functors are interpreted as mapping each ordered pair of the UD to a single member, three-place functors are interpreted as mapping each ordered triple of the UD to a single member, etc). It is true on the following interpretation.

UD: {PI}

Px: x is even

Hx: x is odd

*f*(x): the successor of x

(i.e., the number that results from adding 1 to x).

- The same sentence '$(\forall x)(Px \supset Hf(x))$', however, is false on the following interpretation.

UD: {PI}

Px: x is even

Hx: x is odd

$f(x)$: the square of x

- 'The square of any even positive integer is an odd positive integer' is false.

`($\forall$x)($\forall$y)$f$(x,y) = $f$(y,x)' is true on this interpretation:

UD: {PI}

*f(x,y):* the sum of x and y

since the sum of any two positive integers x and y is equal to the sum of y and x. However, the sentence is false on this interpretation:

UD: {PI}

*f*(x,y): x raised to the power y

since it is not true that x raised to the power of y equals y raised to the power of x.

- Remember, functors are functions, and as such cannot be interpreted as a one-to-many mapping. E.g. `$f$(x)' cannot be interpreted as `the integer greater than x' since there is always more than one greater integer for any x (in fact, there's always an infinite number!).

- Also, interpretations of functions must meet two conditions: 1. A one-place (or n-place) functor must be defined for the entire UD. E.g. `$f$(x) can't mean `intergral square root of x' since not every integer has an integral square root; 2. The value of the function must be in the UD. E.g. `$f$(x,y)' can't mean `the result of dividing y by $x$' since there are results not in the positive integers.

- '$(\forall x)(\exists y)\ y = f(x)$' is quantificationally true. Since `$f$' must meet the conditions specified, it is true on any interpretation that, if for each x there is at least one y, then the pair x and y satisfies `$y = f(x)$'.

$(\forall x)\ Pf(x)$
$Pf(f(a))$

is quantificationally invalid, since there is an interpretation that makes the premise true but makes the conclusion false. E.g. UD: {PI}; Px:x is greater than or equal to 3; $f(x)$: the successor of x; a: 1.

- Sentences containing functors can also be truth-functionally expanded, but as with identity, certain conditions must be met to avoid complications. In addition to the two conditions we've covered, we also have the following third condition:

3. For each **n**-place functor $f$ occurring in one or more of the sentences being expanded and each sequence of **n** constants $a_1$, ..., $a_n$ from the set of constants { $b_1$, ..., $b_m$ } for which the sentence(s) is (are) being expanded, the sentence (...($f$ ($a_1$, ..., $a_n$) = $b_1 \vee f$ ($a_1$, ..., $a_n$) = $b_2$ ) $\vee$ ... $\vee f$ ($a_1$, ..., $a_n$) = $b_m$ ) must be true.

- In other words, the value of applying the function to $a_1$, ..., $a_n$ must be named by one of the constants in the set of constants used for the expansion.

| Nba | Nbb | I | (Naa | & | Nba) | ∨ | (Nab | & | Nbb) |
|---|---|---|---|---|---|---|---|---|---|
| F | T | I | T | F | F | T | T | T | T |

- E.g. the sentence `$(\forall x)(Px \supset Pf(x))$' cannot be false on any interpretation with a one-member UD. But if we don't observe the above conditions, then the truth functional expansion, `$Pa \supset Pf(a)$', produces a row in the truth-table on which it is false, which would be a mistake!

| Pa | P $f$(a) | | Pa | $\supset$ | P $f$(a) |
|----|----------|----|----|----|----|
| **T** | **F** | **I** | **T** | **F** | **F** |

- E.g. In the next slide we construct a truth-table for the truth-functional expansion of '$(\forall x)(Px \supset Pf(x))$' for constants {'a','b'}. In order to satisfy condition 3, we add two sentences, which appear to the right of the vertical line

$f$(a)=a   $f$(a)=b   $f$(b)=a   $f$(b)=b   Pa   Pb   P$f$(a)   P$f$(b)

$\downarrow$         $\downarrow$

(Pa⊃P$f$(a))  &  (Pb⊃P$f$(b))      $f$(a)=a  ∨  $f$(a)=b    $f$(b)=a  ∨  $f$(b)=b

**T**          **T**

- Again, the reason why we assign T to both disjunctions is because of condition 3.

Let us assign some truth values

| $f$(a)=a | $f$(a)=b | $f$(b)=a | $f$(b)=b | Pa | Pb | P $f$(a) | P $f$(b) |
|----------|----------|----------|----------|----|----|----------|----------|
| T | F | F | T | | | | |

By condition 2, `Pa' and `P$f$(a)', must have the same truth-value because we made `$f$(a)=a', and the same goes with `Pb' and `P$f$(b)' because we made `$f$(b)=b' true.

| (Pa⊃P$f$(a)) & (Pb⊃P$f$(b)) | $f$(a)=a | ∨ | $f$(a)=b | $f$(b)=a | ∨ | $f$(b)=b |
|---|---|---|---|---|---|---|
| | T | T | F | F | T | T |